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On correspondence between right neardomains and sharply 2–transitive groups

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Abstract. The right neardomain is defined to loosen neardomain axioms. Correspondence of a class of the right neardomains and a class of sharply 2–transitive groups is constructed.

Keywords: neardomain, sharply 2–transitive groups.

In [1, 2] for exposition of *sharply 2–transitive groups* the concept *neardomain* is introduced. Neardomain is an algebraic system $(B, \cdot, +, ^{-1}, 0)$ with two binary operations $\cdot, +$ for which axioms hold:

1. $(B, +)$ is a loop with a unit element 0;
2. $a + b = 0 \Rightarrow b + a = 0$;
3. $(B_1, \cdot, ^{-1})$ is a group with an unit element e, where $B_1 = B \setminus \{0\}$;
4. $(\forall x \in B) \quad x \cdot 0 = 0$;
5. $(\forall x, y, z \in B) \quad (x + y) \cdot z = x \cdot z + y \cdot z$;
6. $(\forall a, b \in B)(\exists\, r_{a,b} \in B_1) \quad (x + a) + b = x \cdot r_{a,b} + (a + b)$ for any $x \in B$.

Until recently it is not known any example of a neardomain which is not a nearfield. In the given work it is offered to loosen neardomain axioms, having left only necessary ones for construction of sharply 2–transitive groups. In particular, it is offered to refuse from axioms 2, 4 and to loosen axioms 1, 5.

Let's define the right neardomain as algebraic system $(B, \cdot, +, -, ^{-1}, 0)$ with operations: $(+) : B \times B_1 \to B$, $(-) : B \times B_1 \to B$, $(\cdot) : B \times B_1 \to B$, where $B_1 = B \setminus \{0\}$, for which axioms are hold

A1. $(\forall x \in B)(\forall y \in B_1)\ (x - y) + y = x$;
A2. $(\forall x \in B)(\forall y \in B_1)\ (x + y) - y = x$;
A3. $(\forall x \in B_1)\ \ x - x = 0$;
A4. $(B_1, \cdot, ^{-1})$ is a group with an unit element $e \in B_1$;
A5. $(\forall x \in B)(\forall y, z \in B_1)(\exists\, h(y, z) \in B_1)\ (x + y)z = xh(y, z) + yz$;
A6. $(\forall x \in B)(\forall y, z \in B_1 :\ y + z \neq 0)(\exists\, r(y, z) \in B_1)\ (x + y) + z = xr(y, z) + (y + z)$;
A7. $(\forall x \in B)(\forall z \in B_1)(\exists\, v(z) \in B_1)\ (x + (0 - z)) + z = xv(z)$.

Axioms A1—A3 define algebraic system $(B, +, -, 0)$ as the right loop. We define labels $L(x) = 0 - x$ then from A1 follows $L(x) + x = 0$. Thus the map $L : B_1 \to B_1$ defines left inverse in the right loop.

Let's consider now the elementary consequences of axioms.

Lemma. *In the right neardomain the following properties hold:*
1. $(\forall x \in B_1)\ 0x = 0$*;*
2. $h(x, y) = EL(x)L(xy)$*, where* $E(x) = x^{-1}$*,* EL *— superposition of transformations* L *and* E*;*
3. $r(y, z) = E(L(z) - y)L(y + z)$*;*

4. $x - z = xEv(z) + L(z)$;
5. $v(z) = EL^2(z)z$

Let's define a map $u : B_1 \to B$ by the rule $u(x) = 0x$.

From A5 follows, that $(\forall x, y \in B_1)$ $(L(x) + x)y = L(x)h(x, y) + xy = u(y)$, hence

$$h(x, y) = EL(x)(u(y) - xy). \tag{1}$$

If we will sequentially apply A5 for arbitrary $z, t \in B_1$ then we receive:

$$h(y, z)h(yz, t) = h(y, zt).$$

Let's write the given equality applying the expression (1). With the reduction account, we will receive equality : $(u(z)-yz)EL(yz) = e$, hence, $u(z) = L(yz)+yz = 0$. Thus, the first and second conditions of the lemma are satisfied.

Let's consider now consequences from A6. Let $x = L(y + z)(r(y, z))^{-1} \Rightarrow (L(y + z)(r(y, z))^{-1} + y) + z = 0$, whence we will receive the expression from the third condition of the lemma.

In the case when $y + z = 0$, we will consider consequences from A7 and A2: $x + L(z) = xv(z) - z$. We define $x' = xv(z)$, hence the fourth condition of the lemma is fulfilled $x'E(v(z)) + L(z) = x' - z$.

Let's note A2 with the condition of the received expression $(x + z) - z = (x + z)Ev(z) + L(z) = x$. At $x = 0$ we will receive equality $zEv(z) + L(z) = 0$. Then with the account $L^2(x) = LL(x)$, we will come to justice of the fifth condition of the lemma. □

The operation "$-$" is expressed through the operations "$+$", "$\cdot$", L, E hence we will understand algebraic system $(B, \cdot, +, -, ^{-1}, 0)$ as $(B, \cdot, +, ^{-1}, L, 0)$.

Let's consider the algebraic system $(H, \cdot, \phi, ^{-1}, 0)$ from [3], with the operations:

$$(\cdot) : H \times H_1 \to H, \phi : H \to H, \text{ where } H_1 = H \setminus \{0\},$$

for which the following axioms are fulfilled:

F1. $(H_1, \cdot, ^{-1})$ is a group with an unit element e;
F2. $0x = 0, x \in H_1$;
F3. $\phi(e) = 0$;
F4. $\phi(\phi(x)\phi(y)) = \phi(x\phi(y^{-1}))y, x \in H, y \in H_1 \setminus \{e_1\}$,

The similar algebraic system was investigated in [4].

Theorem 1. *The class of algebraic systems* $(B, \cdot, +, ^{-1}, L, 0)$ *and* $(B, \cdot, ^{-1}, \phi, 0)$ *are rational equivalent.*

Let's introduce a map $\phi : B \to B$, defined in an aspect $\phi(x) = x(0 - e) + e = xa+e$. Let's calculate quadrate of function ϕ taking into account the conditions two and five of the lemma:

$$\phi^2(x) = (xa + e)a + e = (xL(a) + a) + e = xL(a)EL^2(e) = x.$$

From the definition follows $\phi(e) = a + e = 0$ and $\phi(0) = e$. By means of the map ϕ it is possible to express additive operation. Really, $\phi(x)y = (xa + e)y = xL(y) + y$, hence, if $x = zEL(y)$, then $z + y = \varphi(zEL(y))y$. Let's rewrite now identity from A2: $z = (z + y) - y = \phi(zEL(y))y - y$. Having introduced labels $t = \phi(zEL(y))y$, we express $z = \phi(ty^{-1})L(y)$, then $t - y = \phi(ty^{-1})L(y)$.

Calculating the value $t = (x + z) - (y + z)$ in the case $y \neq L(z)$, useing at first A2: $(x + z) = t + (y + z)$, and then the third identity of the lemma: $(x + z) = (t(r(y, z))^{-1} + y) + z$. Applying twice identity from A2, we have the identity:

$$(x + z) - (y + z) = (x - y)(L(z) - y)^{-1}L(y + z).$$

Let's rewrite the given identity with the account $y \neq e, z = L^{-1}(e)$ replacing additive binary operations by their expressions through the function ϕ:

$$\phi(\phi(x)E\phi(y)) = \phi(xy^{-1})E\phi E(y) = \phi(xy^{-1})E\phi E(y). \tag{2}$$

At $x = 0$ this identity takes a simple form $\phi E\phi(y) = E\phi E(y)$, using it, we note identity (2) for $y = E\phi E(t)$:

$$\phi(\phi(x)\phi(t)) = \phi(x\phi E(t))t. \tag{3}$$

Thus, we have the map $\mathbb{A} : (B, \cdot, +, ^{-1}, L, 0) \to (B, \cdot, ^{-1}, \phi, 0)$.

Let's make the inverse construction. We will consider expression from F4 at $x = e, y = t^{-1}$, then under condition F2 and F3 we come to equality $\varphi^2(t) = \varphi(0)t$. On one hand $\varphi^4(t) = (\varphi(0))^2t$, and on the other hand $\varphi^4(t) = \varphi(\varphi^2(\varphi(t))) = \varphi(\varphi(0)\varphi(t))$. It is also possible to note the last expression with the account F4 and F2: $\varphi(\varphi(0)\varphi(t)) = \varphi(0\varphi(t^{-1}))t = \varphi(0)t$. Thus, we come to equality $\varphi^2(0) = \varphi(0)$, hence, $\varphi(0) = e$ and $\varphi^2(t) = t$.

From F4 for $x = E\varphi E(y)$ follows, that $(\forall y \in B_1 \setminus \{e\})\ \varphi E\varphi(y) = E\varphi E(y)$.

By means of arbitrary bijection $L : B_1 \to B_1$ we introduce operations

$$x + y = \varphi(xEL(y))y,\ x - y = \varphi(xy^{-1})L(y).$$

With the account of F2, F3 and $\varphi^2 = id$ it is easy to check up the performance of the axioms A1—A3 of the right loop. The performance A5 follows from the operation definition:

$$(x + y)z = \varphi(xEL(y))yz = \varphi(xEL(y)L(yz)EL(yz))yz = xEL(y)L(yz) + yz.$$

Then we take advantage of identities $\varphi^2 = id$, $\varphi E\varphi = E\varphi E$ and F4 to receive A6:

$$(x + y) + z = \varphi(\varphi(xEL(y))yEL(z))z =$$
$$\varphi(xEL(y)\varphi E\varphi(yEL(z)))\varphi(yEL(z))z =$$
$$\varphi(xEL(y)\varphi E\varphi(yEL(z))L[\varphi(yEL(z))z]EL[\varphi(yEL(z))z])\varphi(yEL(z))z =$$
$$xEL(y)\varphi E\varphi(yEL(z))L[\varphi(yEL(z))z] + (y + z) =$$
$$xE(\varphi(L(z)E(y))L(y))L[\varphi(yEL(z))z] + (y + z) = xE(L(z) - y)L(y + z) + (y + z).$$

Now we take advantage of identity $\varphi^2 = id$ for construction of expression A7:

$$(x + L(z)) + z = \varphi(\varphi(xEL^2(z))L(z)EL(z))z = \varphi^2(xEL^2(z))z = xEL^2(z)z.$$

For any bijection L we have constructed the map $\mathbb{F}_L : (B, \cdot, ^{-1}, \varphi, 0) \to (B, \cdot, +, ^{-1}, L, 0)$ so, that the algebraic systems $(B, \cdot, ^{-1}, \varphi, 0)$ and $\mathbb{A} \circ \mathbb{F}_L(B, \cdot, ^{-1}, \varphi, 0)$ are isomorphic. In the opposite direction the algebraic systems $(B, \cdot, +', ^{-1}, L', 0)$ and $\mathbb{F}_L \circ \mathbb{A}(B_1, 0, L', \cdot, +')$ are isomorphic only at $L = L'$. $\square$

The group $T_2(B)$ of transformations of a set B is called sharply 2–transitive group, if for arbitrary pairs $(x_1, x_2) \neq (y_1, y_2) \in \widehat{B^2}$, where $\widehat{B^2} = B^2 \setminus \{(x, x) | x \in B\}$ there exists an unique element $g \in T_2(B)$ for which the equalities $g(x_1) = y_1$ and $g(x_2) = y_2$ are held.

Theorem 2. *The class of algebraic systems* $(B, \cdot, ^{-1}, \varphi, 0)$ *and the class of sharply 2–transitive groups* $T_2(B)$ *are rational equivalent.*

On the set $\widehat{B^2}$ we define a function $f : B \times \widehat{B^2} \to B$ as

$$f(x, y_1, y_2) = \varphi(x\varphi(y_1 y_2^{-1}))y_2, \tag{4}$$

if $y_2 \neq 0$ and $f(x, y_1, 0) = xy_1$ otherwise. Not to consider two cases separately, we, by means of multiplicative partial operation $(\cdot) : B \times B_1 \to B$, define the groupoid on B so, that $(\forall x \in B)\ x0 = \varphi(x)$, $0^{-1} = 0$.

Let's define a binary operation G on the set $\widehat{B^2}$ in the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} f(x_1, y_1, y_2) \\ f(x_2, y_1, y_2) \end{pmatrix} = \begin{pmatrix} \varphi(x_1\varphi(y_1 y_2^{-1}))y_2 \\ \varphi(x_2\varphi(y_1 y_2^{-1}))y_2 \end{pmatrix}. \tag{5}$$

Supposing, that there are pairs $(x_1, x_2) \neq (y_1, y_2) \in \widehat{B^2}$, that $f(x_1, y_1, y_2) = f(x_2, y_1, y_2)$. Then, for $y_2 \neq 0$ after multiplication of the both parts of equality on y_2^{-1} and transformations by the function φ, we will come to equality $x_1\varphi(y_1 y_2^{-1}) = x_2\varphi(y_1 y_2^{-1})$ from which follows, that $x_1 = x_2$. At $y_2 = 0$ we get the equality $x_1 y_1 = x_2 y_1$, hence, $x_1 = x_2$. We have come to an inconsistency. Thus, the operation G, defined above, is a groupoid.

It is easy to check, that the pair $(e, 0) \in \widehat{B^2}$ is the left and the right unit element. Now we check the associativity:

$$\varphi(\varphi(x_i\varphi(y_1 y_2^{-1}))y_2\varphi(z_1 z_2^{-1}))z_2 = \varphi(\varphi(x_i\varphi(y_1 y_2^{-1}))\varphi\varphi(y_2\varphi(z_1 z_2^{-1})))z_2 =$$
$$\varphi(x_i\varphi(y_1 y_2^{-1})\varphi E\varphi(y_2\varphi(z_1 z_2^{-1})))\varphi(y_2\varphi(z_1 z_2^{-1}))z_2 =$$
$$\varphi(x_i\varphi(y_1\varphi(z_1 z_2^{-1})E\varphi(y_2\varphi(z_1 z_2^{-1})))\varphi(y_2\varphi(z_1 z_2^{-1}))z_2.$$

We have come to a semigroup with a unit element. We will discover now the left inverse:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \varphi(x_2^{-1})E\varphi(x_1 x_2^{-1}) \\ E\varphi(x_1 x_2^{-1}) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} e \\ 0 \end{pmatrix}.$$

Let's check that it is also the right inverse:

$$\varphi(x_i\varphi(\varphi(x_2^{-1})E\varphi(x_1 x_2^{-1})\varphi(x_1 x_2^{-1})))E\varphi(x_1 x_2^{-1}) = \varphi(x_i x_2^{-1})E\varphi(x_1 x_2^{-1}).$$

Thus, we defined that G is a group, but since it operates on the set $\widehat{B^2}$ sharply transitive, then the group G at an operation on the set B it will be sharply 2–transitive, hence, we have constructed the map $\mathbb{G} : (B, \cdot, ^{-1}, \varphi, 0) \to T_2(B)$.

Now we make the inverse construction and on a group $T_2(B)$ we will construct an algebraic system $(B, \cdot, ^{-1}, \phi, 0)$. For an arbitrary pair $(e_1, e_2) \in \widehat{B^2}$ it is possible to construct the bijective map $T_2(B) \to \widehat{B^2}$, putting in correspondence to an element $g \in T_2(B)$ the pair $[x_1, x_2]$ so that

$$(e_1, e_2) \cdot g = (e_1, e_2) \cdot [x_1, x_2] = (e_1 \cdot [x_1, x_2], e_2 \cdot [x_1, x_2]) = (x_1, x_2). \quad (6)$$

The given bijection induces the isomorphic group $G \simeq T_2(B)$ on the set of pairs $\widehat{B^2}$. The pair $[e_1, e_2]$ is an unit of the group G.

At serial transformation of the pair (e_1, e_2) by elements $[x_1, x_2]$ and $[y_1, y_2]$ we come to equality:

$$[x_1, x_2][y_1, y_2] = [x_1 \cdot [y_1, y_2], x_2 \cdot [y_1, y_2]], \quad (7)$$

from which, with the account (6), follows, that on a subset $B_1 = \{x \in B | [x, e_2] \in G\}$ it is possible to introduce the group structure naturally. The map $e_1 \cdot [x, e_2] \mapsto x$ induces on B_1 a group structure. Multiplication in the group B_1, as well as in the group $T_2(B)$ we will write without a point. We will expand the group operation to a partial operation $B \times B_1 \to B$, having predetermined it in an aspect $e_2 y = e_2 \cdot [y, e_2] = e_2$ so, that e_2 will be the left zero in a partial operation $(\cdot) : B \times B_1 \to B$.

From (6) and (7) follows,$[e_2, e_1]$ is an involution of G. We define $\phi : B \to B$ in an aspect $\phi(x) = x \cdot [e_2, e_1]$, then $\phi(e_1) = e_2$ and

$$[e_2, e_1][x_2, x_1] = [x_1, x_2] = [\phi(x_1), \phi(x_2)][e_2, e_1]. \quad (8)$$

For an arbitrary $[e_1, x_2] \in G$, at $x_2 \in B_1 \setminus \{e_1\}$ it is possible to note:

$$[e_1, x_2] = [x_2^{-1}, e_1][x_2, e_2] = [\phi(x_2^{-1}), e_2][e_2, e_1][x_2, e_2].$$

On the other hand, with the account (8) for $[e_1, x_2]$ it is fair

$$[e_1, x_2] = [e_2, e_1][\varphi(x_2), e_2][e_2, e_1].$$

Having taken advantage of the two received expressions and equating outcomes of transformation arbitrary $t \in B$ by element $[e_1, x_2] \in G$, we come to identity:

$$\phi(\phi(t)\phi(x_2)) = \phi(t\phi(x_2^{-1}))x_2, \ t \in B, x_2 \in B_1 \setminus \{e_1\}.$$

The map $\mathbb{F}_{(e_1,e_2)} : T_2(B) \to (B, \cdot, ^{-1}, \phi, 0)$ is constructed, putting in correspondence to group $T_2(B)$ algebraic system $(B, \cdot, ^{-1}, \phi, 0)$.

Let's notice still, that for arbitrary $[x_1, x_2] \in T_2(B)$ it is possible to note:

$$[x_1, x_2] = \begin{cases} [\phi(x_1 x_2^{-1}), e_2][e_2, e_1][x_2, e_2], & x_2 \in B_1, \\ [x_1, e_2], & x_2 = e_2. \end{cases}$$

Then for arbitrary $t \in B$ under condition of $x_2 \neq e_2$ and $t \cdot [x_1, e_2] = t x_1$ the equality:

$$t \cdot [x_1, x_2] = t \cdot [\phi(x_1 x_2^{-1}), e_2][e_2, e_1][x_2, e_2] = \phi(t\phi(x_1 x_2^{-1}))x_2 \quad (9)$$

is fair. Comparing (4), (5) with (9) and (7) we come to that there is a natural isomorphism $\mathbb{G} \circ \mathbb{F}_{(e_1,e_2)} : T_2(B) \to T_2'(B)$, thus $\mathbb{G} \circ \mathbb{F}_{(e_1,e_2)} = id$. Isomorphism of algebraic systems $\mathbb{F}_{(e_1,e_2)} \circ \mathbb{G} : (B, \cdot, ^{-1}, \phi, 0) \to (B', \cdot', ^{-1}, \phi, e_2)$ is set by map $\mathbb{F}_{(e_1,e_2)} \circ \mathbb{G} : x \mapsto \varphi(x\varphi(e_1 e_2^{-1}))e_2$, thus $\mathbb{F}_{(e_1,e_2)} \circ \mathbb{G} = id$. $\square$

Let's consider some examples of the right neardomains constructed over a skew field $\mathbb{K}$ for which $\varphi(x) = -x + 1$, $x \in \mathbb{K}$. As the first example we consider $L(x) = ax$:

$$x \oplus y = -xa^{-1} + y,\ x \ominus y = -xa + ay,\ r(y, z) = -a^{-1},\ v(z) = a^{-2}.$$

In such right neardomain bilaterial distributivity is fulfilled and the identity $L(x \oplus y) = L(x) \oplus L(y)$ is hold. For the second example over a skew field we consider $L(x) = -x^{-1}$, then

$$x \oplus y = xy^2 + y,\ x \ominus y = xy^{-2} - y^{-1},\ r(y, z) = y^2 z(z+y)^{-1}(yz+1),\ h(y, z) = z^{-1}.$$

For the given right loop $L(x \oplus y) \neq L(x) \oplus L(y)$, but it is fulfilled $L(x) \oplus x = x \oplus L(x) = 0$.

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